Additivity in Classical and Quantum Shannon Theory

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Joint work with
Andrew Cross and Ke Li

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Information theory: sending, storing, processing data

\[ \rho_{AB} \]

<table>
<thead>
<tr>
<th>Alice</th>
<th>Bob</th>
</tr>
</thead>
</table>

\[ |\phi_2\rangle_{AB} \]

\[ E \rightarrow D \]

Shared Secret Key

Eve, the Eavesdropper
Entropies quantify the answers

- \( H(X) = - \sum_x p_x \log p_x \)
- \( H(\rho) = -\text{Tr} \ \rho \log \rho \)

- Optimal Compression: \( H(X) \)
- Schumacher Compression: \( H(\rho) \)
- Classical Channel capacity: \( \max I(X;Y) \)
  \[ I(X;Y) = H(X) + H(Y) - H(XY) \]
- Quantum Communication: \( H(B) - H(E) \)
- Private capacity: \( I(V;B)-I(V;E) \)
- Strong Subadditivity:

\[ I(A;B|C) = H(AC) + H(BC)-H(ABC)-H(C) \geq 0 \]
Additivitity lets us calculate answers

\[ C(\quad) \otimes \quad = \ C(\quad) + C(\quad) \]

Classical Capacity of Classical Channel
Nonadditivity is the rule
Especially quantumly

Blessing: Better rates for classical and quantum communication.

Curse:
Mostly don’t know capacities, distillable entanglement, etc.
Have upper and lower bounds that are far apart.
Noisy Quantum Channels

• Noiseless quantum evolution: \( \rho \rightarrow U \rho U^\dagger \)
  
  Unitary satisfies \( U^\dagger U = I \)

• Noisy quantum evolution: unitary interaction with inaccessible environment

\[
f(\mathcal{N}) = \max_{\phi V_1 \ldots V_n A} f(\mathcal{N}, \phi V_1 \ldots V_n A)
\]
Untangling Additivity

\[ f(\mathcal{N}) = \max_{\phi_{V_1 \cdots V_n A}} f(\mathcal{N}, \phi_{V_1 \cdots V_n A}) \]

\[ f(\mathcal{N}, \phi_{V_1 \cdots V_n A}) = \sum_{s \in \mathcal{P}(V_1 \cdots V_n \text{BE})} \alpha_s H(s) \]

Basically all additivity proofs do this:
Take \( \phi_{12} \) that gives a value \( f(\mathcal{N} \otimes \mathcal{M}, \phi_{12}) \)

1) Decoupling: \( \phi_{12} \rightleftharpoons \phi_1 \)

2) Apply strong subadditivity to show:
\[ f(\mathcal{N} \otimes \mathcal{M}, \phi_{12}) \leq f(\mathcal{N}, \phi_1) + f(\mathcal{M}, \phi_2) \]

Goal: classify entropic additive formulas of this sort “uniformly additive”
All uniformly additive entropy formulas

“All the Formulas That’s Fit to Print”

One Auxiliary Variable:

<table>
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<tr>
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Rest of talk: explain how to get this table and what it means.
Outline

- Standard Additivity Proofs
- Results: all uniformly additive formulas
- Completely coherent information: a new additive quantity
- Further Observations
Canonical example: Entanglement Assisted Capacity

• Entanglement assisted capacity:

$$C_{ea}(\mathcal{N}) = \max_{\phi_{VA}} I(V;B)$$

with $$I(V;B) = H(V)+H(B)-H(VB).$$

$$C_{ea}(\mathcal{N} \otimes \mathcal{M}) = C_{ea}(\mathcal{N}) + C_{ea}(\mathcal{M})$$

Proof:
Let $$\phi_{VA_1A_2}$$ be optimal for $$C_{ea}(\mathcal{N} \otimes \mathcal{M}).$$

$$I(V; B_1B_2) = I(V; B_1) + I(V; B_2|B_1)$$

$$= I(V; B_1) + I(VB_1; B_2) - I(B_1; B_2)$$

$$\leq I(V; B_1) + I(VB_1; B_2) \leq C_{ea}(\mathcal{N}) + C_{ea}(\mathcal{M})$$
Two key steps

- Decoupling

\[
\phi_{V_{A_1 A_2}} \quad \Rightarrow \quad \hat{\phi}_{V A_2} = \phi_{V B_1 | A_2}
\]

\[
\tilde{\phi}_{V A_1} = \phi_{V A_1}
\]

- Entropy inequality

\[
I(V; B_1 B_2) = I(V; B_1) + I(V; B_2 | B_1)
\]

\[
= I(V; B_1) + I(V B_1; B_2) - I(B_1; B_2)
\]

\[
\leq I(V; B_1) + I(V B_1; B_2) \leq C_{ea}(\mathcal{N}) + C_{ea}(\mathcal{M})
\]
Standard Decouplings

\[ \phi_{VA_1A_2} \]

\[ \tilde{\phi}_{\bar{V}A_1} \]

\[ \phi_{\bar{V}A_2} \]
Standard Decouplings

\[ \tilde{\phi}_{V A_1} \]

What to do with \( A_2 \)?
Standard Decouplings

What to do with $A_2$?
Standard Decouplings

\[ \phi_{\tilde{V}A_1} \]

\[ \phi_{V A_1 A_2} \]

\[ \phi_{\tilde{V} A_2} \]

Make \( \tilde{V} \) out of these
Standard Decouplings

\[ \phi_{VA_1A_2} \rightarrow \tilde{\phi}_{\tilde{V}A_1} \]

\[ \phi_{VA_1A_2} \rightarrow \hat{\phi}_{\hat{V}A_2} \]

Diagram:

- \( \phi_{VA_1A_2} \)
- \( V \)
- \( A_1 \)
- \( A_2 \)
Standard Decouplings

\[ \phi_{\hat{V}A_1} \rightarrow \phi_{\hat{V}A_1} \]

\[ \phi_{VA_1 A_2} \rightarrow \phi_{\hat{V}A_2} \]

\[ \hat{\phi}_{\hat{V}A_1} \]

\[ \hat{\phi}_{\hat{V}A_2} \]

\[ V \]

\[ B_2 \]

\[ E_2 \]

Make out of these \( \hat{V} \)
Two key steps

• Decoupling

\[ \hat{\phi}_{VA_2} = \phi_{VB_1|A_2} \]

\[ \tilde{\phi}_{VA_1} = \phi_{VA_1} \]

• Entropy inequality

\[
I(V; B_1B_2) = I(V; B_1) + I(V; B_2|B_1) \\
= I(V; B_1) + I(VB_1; B_2) - I(B_1; B_2) \\
\leq I(V; B_1) + I(VB_1; B_2) \leq C_{ea}(N) + C_{ea}(M)
\]
Entropic Inequalities

Strong subadditivity:

\[ I(A;B|C) = H(AC) + H(BC) - H(ABC) - H(C) \geq 0 \]

Also:

\[ H(A|C) + H(A|D) \geq 0 \]

There may be more, but we don’t know them.
Entropy inequalities

- For $n$ systems $A_1 \ldots A_n$, there are $2^n - 1$ different entropies:
  $$(H(A_1), \ldots, H(A_n), H(A_1 A_2)\ldots, H(A_1 \ldots A_n))$$

- Not all vectors can be realized. Realizable vectors form a cone.
- Cone bounded by various planes defined by strong subadditivity.
- Can test if an identity is satisfied by Linear Programming.
- There are other “non-Shannon” classical inequalities, but for quantum we don’t know (though there probably are).
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Entropic formulas

\[ f_\alpha(\mathcal{N}, \phi_{V_1...V_nA}) = \sum_{s \in \mathcal{P}(V_1...V_nBE)} \alpha_s H(s) \]

Fix a standard decoupling \( \phi_{V_1A_1A_2} \rightarrow (\tilde{\phi}_{\tilde{V}_1A_1}, \hat{\phi}_{\hat{V}_1A_2}) \)

\[
\Delta \left( f_\alpha, U_{N_1}, U_{N_2}, \phi_{V_1...V_nA_1A_2}, \tilde{\phi}_{\tilde{V}_1...\tilde{V}_nA_1}, \hat{\phi}_{\hat{V}_1...\hat{V}_nA_2} \right) = \\
f_\alpha(U_{N_1}, \tilde{\phi}_{\tilde{V}_1...\tilde{V}_nA_1}) + f_\alpha(U_{N_2}, \hat{\phi}_{\hat{V}_1...\hat{V}_nA_2}) - f_\alpha(U_{N_1} \otimes U_{N_2}, \phi_{V_1...V_nA_1A_2})
\]

Find \( \alpha \) such that \( \Delta(f_\alpha) \geq 0 \) for all \( N_1, N_2, \phi_{V_1...V_nA_1A_2} \).
Zero Auxiliary Variables

\[ f_\alpha(N, \phi_A) = \alpha_B H(B) + \alpha_E H(E) + \alpha_{BE} H(BE) \]

Decoupling: \( \phi_{A_1 A_2} \rightarrow (\phi_{A_1}, \phi_{A_2}) \)

\[ \Delta(\alpha, \phi_{A_1 A_2}) = f_\alpha(N_1, \phi_{A_1}) + f_\alpha(N_2, \phi_{A_2}) - f_\alpha(N_2, \phi_{A_1 A_2}) \]
\[ = \alpha_B I(B_1; B_2) + \alpha_E I(E_1; E_2) + \alpha_{BE} I(B_1 E_1; B_2 E_2) \]

When is \( \Delta(\alpha, \phi_{A_1 A_2}) \geq 0? \)
Zero Auxiliary Variables

When is $\Delta(\alpha, \phi_{A_1A_2}) \geq 0$?

**Rays**

$$f_\alpha = \lambda_1 H(B) + \lambda_2 H(E) + \lambda_3 H(B|E) + \lambda_4 H(E|B)$$

$$\lambda_i \geq 0$$

**Faces**

$$\alpha_B + \alpha_{BE} \geq 0$$
$$\alpha_E + \alpha_{BE} \geq 0$$
$$\alpha_B + \alpha_E + \alpha_{BE} \geq 0$$
$$\alpha_{BE} \geq 0.$$

Anything inside the cone is uniformly additive.

Outside the cone, there is a state that makes $\Delta < 0$

To show, find state that makes, e.g.,

$$\Delta = \alpha_B + \alpha_{BE}$$
One Auxiliary Variable

Enough to consider

\[ f^V_\alpha (\mathcal{N}, \phi_{VA}) = \alpha_V H(V) + \alpha_{BV} H(BV) + \alpha_{EV} H(EV) + \alpha_{BEV} H(BEV) \]

Fix a standard decoupling:

\[ \tilde{V} \in \{ V, B_2V, E_2V, B_2E_2V \} \quad \text{and} \]

\[ \hat{V} \in \{ V, B_1V, E_1V, B_1E_1V \} \]

These are labeled by \((a, b)\) \(a, b = 0...3\)

Define \(\Delta_{(a,b)}(\alpha^V)\), find \(\alpha^V = (\alpha_V, \alpha_{BV}, \alpha_{EV}, \alpha_{BEV})\) with \(\Delta_{(a,b)} \geq 0\).

This gives a cone of additive functions for each \((a,b)\)
### One Auxiliary Variable

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\[ f^V_{\alpha}(N, \phi_{VA}) = \alpha_V H(V) + \alpha_{BV} H(BV) + \alpha_{EV} H(EV) + \alpha_{BEV} H(BEV) \]
One Auxiliary Variable

Most general:  \( f_\alpha = f_\alpha^\phi + f_\alpha^V \)

With

\[
f_\alpha^\phi = \lambda_1 H(B) + \lambda_2 H(E) \\
+ \lambda_3 H(B|E) + \lambda_4 H(E|B)
\]

\[
f_\alpha^V(N, \phi_{VA}) = \alpha_V H(V) + \alpha_{BV} H(BV) + \alpha_{EV} H(EV) + \alpha_{BEV} H(BEV)
\]
Many Auxiliary Variables

Most general: \( f_\alpha = f_\alpha^\phi + f_\alpha^{V_1} + f_\alpha^{V_2} + f_\alpha^{V_1 V_2} \)

Just pick each individual function from the appropriate row in the table.
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## Completely Coherent Information

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<td>$\alpha_{BEV} \geq 0$</td>
<td>$\pm [H(EV) - H(BV)]$</td>
</tr>
<tr>
<td>5.</td>
<td>(1,2)</td>
<td>$B_1$</td>
<td>$E_2$</td>
<td>(2,1)</td>
<td>$\alpha_{BEV} \geq 0$</td>
<td>$H(E</td>
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<td>$\alpha_V \geq 0$</td>
<td>$-H(E</td>
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Completely Coherent Information: Properties

\[ I^{cc}(N) = \max_{\phi_{VA}} \left[ H(VB) - H(VE) \right] \]

- Symmetric in B ↔ E
- Lower bound for cost of swapping B and E.
- Upper bound for simultaneous quantum communication rate to B and E
- For degradable channels, \( I^{cc}(N) = Q(N) = Q^{(1)}(N) \)
- WANT: cardinality bound on V
Generalizing Completely Coherent Information

We can replace the entropies in $I^{cc}$ with any function $J(\rho)$ that satisfies:

- $J(\rho_{V1B1} \otimes \rho_{V2B2}) = J(\rho_{V1B1}) + J(\rho_{V2B2})$
- $F_J(N) = \max [J(VB) - J(VE)]$

Furthermore, if $J$ is monotonic under CP maps, $F_J(N)$ doesn’t need auxiliary variable

J could be: renyi entropy, sandwiched entropy, $(\alpha, z)$-entropies…
Connection to Symmetric Side Channels

Quantum capacity with symmetric zero-capacity channel
Additive upper bound for $Q$

\[
Q_{ss}(\mathcal{N}) = \frac{1}{2}[I(V_1; B|V_2) - I(V_1; E|V_2)]
\]
\[
= \frac{1}{2}[H(BV_2) - H(BV_1 V_2) - H(EV_2) + H(EV_1 V_2)]
\]
\[
= \frac{1}{2}[H(BV_2) - H(EV_2) + H(EV_1 V_2) - H(BV_1 V_2)]
\]
Connection to Symmetric Side Channels

Quantum capacity with symmetric zero-capacity channel
Additive upper bound for $Q$

\[ Q_{ss}(\mathcal{N}) = (1/2)[I(V_1; B|V_2) - I(V_1; E|V_2)] \]
\[ = (1/2)[H(BV_2) - H(BV_1V_2) - H(EV_2) + H(EV_1V_2)] \]
\[ = (1/2)[H(BV_2) - H(EV_2) + H(EV_1V_2) - H(BV_1V_2)] \]
Outline

• Standard Additivity Proofs
• Results: all uniformly additive formulas
• Completely coherent information: a new additive quantity
• Further Observations
A Classical-Quantum Coincidence

- You can do this whole game for classical entropic formulas too.
- You get *exactly* the same set of uniformly additive functions.
- Could have been more, since there are more classical inequalities: $H(X|Y) \geq 0$
- But uniform additivity only uses strong subadditivity.
Further Directions

• Classically, there are additive functions that are not uniformly additive (e.g. $H_{\text{min}}$)
• Single-letter entropic constraints that imply additivity of coherent information (informationally degradable). Only one?
• Applies to rate regions of multi-user information theory
• Constraint on states optimized over
Summary

• Additivity simplifies, but is rare
• Typical additivity proof has two steps:
  1) Decoupling 2) Apply entropy inequalities
• Uniform additivity: standard decoupling + entropy inequalities
• Classified all uniformly additive formulas
• Completely coherent information
Open Questions

• Constrained additivity: 1) constraints on channels 2) constraints on states
• Completely coherent information: operational meaning, cardinality bound
• Understand classical-quantum correspondence better. Coincidence?
• Apply to formulas with c-q states ($\chi, P$)? Decoupling is a challenge there.
THANK YOU